Chapter 1

Rooted forests and trees

The notion of a rooted forest should be familiar to the reader. For completeness, we will give formal definitions.

Let N be a finite set. A *rooted forest on* N is defined by a pair (N, p) where p is a function $p : N \longrightarrow N \cup \{\perp\}$ such that there is no positive integer k such that $p^k(x) = x$ for some element $x \in N$.

The elements of N are *nodes* of the forest (also called *vertices* but we sometimes prefer *node* over *vertex* to emphasize these are elements of a rooted forest). A node x such that $p(x) = \perp$ is a *root*. Each forest has at least one root; it is a tree if it contains only one root.

For each nonroot node x, $p(x)$ is called the *parent* of x in the tree and x is called a *child* of $p(x)$, and the ordered pair $xp(x)$ is called the *parent edge* of x and a *child edge* of $p(x)$. The edges of the forest are the parent edges of its nodes. A node with no children is a *leaf*.

A rooted tree has *arity* k if every node has at most k children. A *binary* tree is a rooted tree that has arity two.

We say two nodes are *adjacent* if one is the parent of the other. We say the edge $xp(x)$ is *incident* to the nodes x and $p(x)$. The degree of a node x, written degree(x), is the number of edges incident to x.

The *ancestors* of x are defined inductively: x is its own ancestor, and (if x is not the root) the ancestors of x's parent are also ancestors of x. If x is the ancestor of y then y is a *descendant* of x. We say y is a *proper* ancestor of x (and x is a *proper* descendant of y) if y is an ancestor of x and $y \neq x$. The *depth* of a node is the number of proper ancestors it has.

We say an edge $xp(x)$ is an *ancestor edge* of y if x is an ancestor of y. We say $xp(x)$ is a *descendant edge* of y if $p(x)$ is a descendant of y.

A *subforest*/*subtree* of (N, p) is a forest/tree (N', p') such that N' is a subset of N, and p' is the restriction of p to N'.

Deletion of an edge $\hat{x}p(\hat{x})$ from a rooted forest (N, p) is an operation that

yields the forest (N, p') where

$$p'(x) = \begin{cases} \perp & \text{if } x = \hat{x} \\ p(x) & \text{otherwise} \end{cases}$$

If F is a rooted forest and e is an edge of F then we use $F - \{e\}$ to denote the result of deleting e.

Deletion of a node $\hat{x}$ from a rooted forest (N, p) is an operation that yields the forest $(N - \{\hat{x}\}, p')$ where

$$p'(x) = \begin{cases} \perp & \text{if } p(x) = \hat{x} \\ p(x) & \text{otherwise} \end{cases}$$

If F is a rooted forest and $\hat{x}$ is a node of F then we use $F - \{\hat{x}\}$ to denote the result of deleting $\hat{x}$.

More generally, if S is a set of nodes or a set of edges, $F - S$ denotes the forest obtained from F by deleting every element of S.

For a tree T and a node x of T, the *subtree rooted at* x is the tree obtained from T by deleting every node that is not a descendant of x.

For a forest T and a node x of T, the *root-to-x path* is the sequence $x_0 x_1 \ldots x_k$ where x_0 is the root of T, x_k is x, and x_i is the parent of x_{i+1} for $i = 0, \ldots, k-1$. We denote this path by $T[x]$.

Ancestorhood defines a partial order among nodes of a forest. Given a set S of nodes of a forest, a *rootmost* node of S in the forest is a node v such that no proper ancestor of v is in S. A *leafmost* node of S is a node v such that no proper descendant of v is in S.

Given two nodes u and v of a forest, we say u is *leafward* of v and v is *rootward* of u if u is a descendant of v. A sequence $v_1, \ldots, v_k$ of nodes of the forest is a *leafward* path if v_i's parent is v_{i+1} for $i = 1, \ldots, k-1$.

1.1 Rootward computations

Suppose T is a rooted tree and $w(\cdot)$ is an assignment of weights to the nodes. There is a simple, linear-time algorithm to compute, for each node u, the total weight of all descendants of u:

```
def totalWeight(u):
    return w(u) + ∑{totalWeight(v) : v a child of u}
```

We call this a *rootward computation* since the order of nodes for which it computes results is consistent with the rootward partial order: children before parents. This algorithmic schema, though simple, comes up again and again: in finding separators for trees (in the next section), in algorithms that exploit interdigitating trees in planar graphs (Section 4.5), in processing a breadth-first-search tree (Section 5.4), in dynamic-programming algorithms on trees (Section 14.1) and on graphs of bounded carvingwidth (Section 14.3.1) and bounded branchwidth (Section 14.5.1).

Note that TOTALWEIGHT(u) must iterate through the children of u, whereas the formal definition of a forest provides only a way to go from a node to its parent. For this algorithm to be efficient, therefore, it should be proceeded by another rootward computation in which a table is constructed that maps each node to its children. The latter computation takes only linear time, and enables other rootward computations, such as TOTALWEIGHT, to be carried out in linear time.

1.2 Separators for rooted trees

A *separator* for a tree is a node or edge whose deletion results in trees that are "small" in comparison to the original graph.

Definition 1.2.1. *For a number $0 < \alpha < 1$, a rooted tree, and an assignment $\hat{w}(\cdot)$ of weights to nodes, we say a node v is α-heavy with respect to $\hat{w}(\cdot)$ if $\hat{w}(v)$ is greater than α times the sum of all weights.*

We say v is a leafmost *α-heavy node if it is α-heavy but its children are not.*

Lemma 1.2.2 (Leafmost Heavy Node)**.** *There is a linear-time algorithm that, given a positive number α less than 1, a rooted tree, and an assignment $\hat{w}(\cdot)$ of weights to nodes such that the weight of each node is at least the sum of the weights of its children, outputs a leafmost α-heavy node of the tree.*

Proof. Call the procedure below on the root of T.

```
define LEAFMOSTHEAVYNODE(v):
1     if some child u of v has ŵ(u) > αW,
2           return LEAFMOSTHEAVYNODE(u)
3     else return v
```

By induction on the number of invocations, for every call LEAFMOSTHEAVYNODE(v), we have $\hat{w}(v) > \alpha W$. If v is a leaf then the condition in Line 1 is not satisfied, so the procedure terminates. Let v_0 be the node returned by the procedure. Since the condition in Line 1 did not hold for v_0, every child v of v_0 satisfies $\hat{w}(v) \leq \alpha W$. □

1.2.1 Node separator

Lemma 1.2.3 (Tree Node Separator)**.** *Let T be a rooted tree, and let $w(\cdot)$ be an assignment of weights to nodes. Let W be the sum of weights. There is a linear-time algorithm to find a node v_0 such that every tree in the forest $T - \{v_0\}$ has total weight at most $W/2$.*

Proof. For each node u, define $\hat{w}(u) = \sum\{w(v) \ : \ v \text{ a descendant of } u\}$. Then $\hat{w}(\text{root}) = W$. The values $\hat{w}(\cdot)$ can be computed using a rootward computation as in Section 1.1. Let v_0 be a leafmost 1/2-heavy node. Let $v_1, \ldots, v_p$ be the

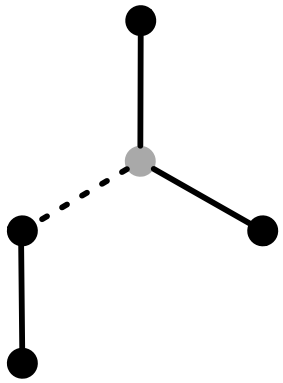


Figure 1.1: A rooted tree. Suppose the nodes are each assigned weight 1. The gray node is a separator whose deletion results in a forest, each of whose trees has weight at most half the total weight. The dashed edge is a separator whose deletion results in a forest, each of whose trees has weight at most three-fourths of the total weight.

children of v_0. For each child v_i, the subtree rooted at v_i has weight at most $W/2$. Each such subtree is a tree of $T - \{v_0\}$. The remaining tree is $T - \{v : v \text{ is a descendant of } v_0\}$. Since the sum $\sum\{w(v) : v \text{ is a descendant of } v_0\} = \hat{w}(v_0)$ exceeds $W/2$, the weight of the remaining tree is less than $W/2$. □

1.3 Edge separators

Lemma 1.3.1 (Tree Edge Separator of Edge-Weight)**.** *Let T be a binary tree, and let $w(\cdot)$ be an assignment of weights to edges. There is a linear-time algorithm to find an edge $\hat{e}$ such that every tree in $T - \{\hat{e}\}$ has at most two-thirds of the weight.*

Proof. Assume for notational simplicity that the total weight is 1. For a nonroot node v, define

$$\hat{w}(v) = w(\text{parent edge of } v) + \sum\{w(e) : e \text{ a descendant edge of } v\}$$

and define $\hat{w}(root) = 1$. Let v_0 be a leafmost 1/3-heavy node with respect to $\hat{w}(\cdot)$. The sum $\sum\{\hat{w}(v) : v \text{ a child of the root}\}$ equals 1. Because the root has at most two children, $\hat{w}(v) \geq 1/2$ for some child of the root. Thus v_9 is not the root of T. Let e_0 be the parent edge of v_0. Then $T - \{e_0\}$ consists of two trees. One tree consists of all descendants of v_0, and the other consists of all nondescendants.

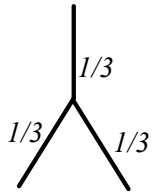
The weight of all edges among the nondescendants is $1 - \hat{w}(v_0)$, which is less than $1 - 1/3$ since $\hat{w}(v_0) > 1/3$. Let $v_1, \ldots, v_p$ be the children of v_0. (Note

that $0 \leq p \leq 2$.) The weight of all edges among the descendants is $\sum_{i=1}^{p} \hat{w}(v_i)$. Since $\hat{w}(v_i) \leq 1/3$ for $i = 1, \ldots, p$ and $p \leq 2$, we infer $\sum_i \hat{w}(v_i) \leq 2/3$. □

The following example shows that the restriction on the arity of the trees in Lemma 1.3.1 cannot be discarded:

If the number of children is k then removal of any edge results in remaining weight $(k-1)/k$.

The following example shows that, for binary trees, the factor two-thirds in Lemma 1.3.1 cannot be improved upon.



For some separators, we need to impose a condition on the weight assignment. We say a weight assignment is *α-proper* if no element is assigned more than an α fraction of the total weight.

Lemma 1.3.2 (Tree Edge Separator of Node Weight)**.** *Let T be a binary tree, and let $w(\cdot)$ be a $\frac{3}{4}$-proper assignment of weights to nodes such that each nonleaf node is assigned at most one-fourth of the weight. There is an edge $\hat{e}$ such that every tree in $T - \{\hat{e}\}$ has at most three-fourths of the weight.*

Proof. Assume the total weight is 1. For each node v, define

$$\hat{w}(v) = \sum \{w(v') \ : \ v' \text{ a descendant of } v\}$$

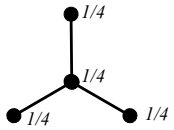
Let v be a leamost 3/4-heavy node with respect to $\hat{w}(\cdot)$. Let $v_1, \ldots, v_p$ be the children of v. Note that $p \leq 2$. Since $w(v) \leq \frac{3}{4}$ but $\hat{w}(v) > \frac{3}{4}$, we must have $p > 0$, so $w(v) \leq \frac{1}{4}$.

For $1 \leq i \leq p$, let W_i be the weight of descendants of v_i. Let $\hat{i} = \text{maxarg}_{1 \leq i \leq p} W_i$. By choice of v, $W_{\hat{i}} \leq \frac{3}{4}$. By choice of $\hat{i}$,

$$W_{\hat{i}} \geq \frac{1}{2} \sum_{i=1}^{p} W_i > \frac{1}{2} \left(\frac{3}{4} - w(v) \right) \geq \frac{1}{2} \left(\frac{3}{4} - \frac{1}{4} \right) = W/4$$

This shows that choosing $\hat{e}$ to be the edge $v_{\hat{i}} v$ satisfies the balance condition. □

The following example shows that the factor three-fourths in Lemma 1.3.2 cannot be improved upon.



By changing our goal slightly, we can get a better-balanced separator.

Lemma 1.3.3 (Tree Edge Separator of Node/Edge Weight)**.** *Let T be a binary tree, and let $w(\cdot)$ be an assignment of weight to the nodes and edges such that, for each node v, the weight assigned to v is at most $\frac{1}{3}(3 - \text{degree}(v))$ times the total weight. There is an edge e such that every rooted tree in $T - \{e\}$ has at most two-thirds of the weight.*

Problem 1.1. *Prove Lemma 1.3.3.*

1.4 Computation time for finding separators

For Lemmas 1.3.1 through 1.3.2, the weight assignment $\hat{w}(\cdot)$ can be obtained from $w(\cdot)$ via a rootward computation, and the linear-time implementation of LeafmostHeavyNode can be employed.

1.4.1 Recursive tree decomposition

In the Appendix, we describe data structures for representing sequences and rooted trees. These can be used to preprocess a rooted tree so as to find recursive separators.

Problem 1.2. *A recursive edge-separator decomposition for a rooted tree T is a rooted tree D such that*

- *the root r of D is labeled with an edge e of T;*
- *for each connected component K of $T - e$ (there are at most two), r has a child in D that is the root of a recursive edge-separator decomposition of K.*

Show that there is an $O(n \log n)$ algorithm that, given a binary tree T with n nodes, returns a recursive edge-separator decomposition of depth $O(\log n)$.

Problem 1.3. *Show that the data structure for representing trees can be used to quickly find edge-separators in binary trees. Use this idea to give a fast algorithm that, given a tree of maximum degree three, returns a recursive edge-separator decomposition of depth $O(\log n)$.* Note: *A running time of $O(n)$ can be achieved.*